An improved multiobjective approach of temporal disaggregation, a special case of multivariate Denton

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1 Introduction

Temporal aggregation is the process of deriving low frequency data from high frequency data. Although the method depends on the type of the data (stock, flow or index), every case can be calculated easily with some linear operations. The yearly GDP, for example, is equal to the sum of the quarterly GDP.
The inverse method which is called temporal disaggregation is much more difficult [1], but sometimes useful in official statistics. It can improve the timeliness of data publication without increasing the reporting burden. In this case a high frequency data is made from low frequency time series, while the temporal aggregation of the high frequency time series must be equal to the known low frequency series. If we need to estimate monthly price index, while we have quarterly observations, the monthly data average should be equal to the quarterly data.
One of the well known solutions of this problem is called the Denton method and its variants. These methods use a known high frequency auxiliary indicator, which helps to find the level or the dynamic of the disaggregated time series. In other words the methods try to minimise the distance between the (first or second differences of) auxiliary indicator and the (first or second differences of) disaggregated time series, while the temporal aggregation of this series should be equal to the known low frequency data. For example, monthly Consumer Price Index may be a good auxiliary indicator to temporal disaggregation of quarterly Producer Price Index.
From optimisation point of view Denton method is a convex quadratic optimisation problem. Let the vector $\mathbf{y} \in \mathbb{R}^T$ denote the high frequency series we are looking for, and let $\tilde{\mathbf{y}} \in \mathbb{R}^t$ be the known low frequency data. The fact that the temporal aggregation condition is fulfilled can be formulated as a linear equation:

$$A\mathbf{y} = \tilde{\mathbf{y}},$$

where $A \in \mathbb{R}^{t \times T}$ is a proper matrix. Let the auxiliary indicator be denoted by $\mathbf{z} \in \mathbb{R}^T$. The Denton problem (DP) can be formulated in the following way:

$$\left.\begin{array}{c} \min\limits_{\mathbf{y}} (\mathbf{y}-\mathbf{z})^\top M(\mathbf{y}-\mathbf{z}) \\ A\mathbf{y} = \tilde{\mathbf{y}} \end{array}\right\} \quad (DP),$$

where $M \in \mathbb{R}^{T \times T}$ is a symmetric positive definite matrix. This matrix depends on whether we like to minimise the distance of the time series or the distance of the first or second difference of the time series.
Sometimes more than one time series need to be disaggregated while one of them is equal to the sum of the others, which fact is called cross-sectional condition. These type of problems are called reconciliation. The multivariate Denton method solves this problem also as a convex quadratic optimisation problem [2]. Let $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R}^T$

be the $k+1$ disaggregated time series we are looking for. Let $\tilde{\mathbf{y}}_0, \tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_k \in \mathbb{R}^t$ be the known low frequency time series, and $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{R}^T$ be the auxiliary indicators. The multivariate Denton problem (MDP) can be formulated in the following way

$$\left.\begin{array}{c} \min\limits_{\mathbf{y}_i} \sum\limits_{i=0}^{k} (\mathbf{y}_i - \mathbf{z}_i)^\top M (\mathbf{y}_i - \mathbf{z}_i) \\ A\mathbf{y}_i = \tilde{\mathbf{y}}_i \quad i = 0, 1, 2, \ldots, k \\ \sum\limits_{i=1}^{k} \mathbf{y}_i - \mathbf{y}_0 = \mathbf{0} \end{array}\right\} \quad (MDP),$$

where matrix A and M are the same as mentioned before.
From the formula of (MDP) it can be seen that the sum of the objective functions is minimised in this case. On the other hand the reconciliation problem can be formulated as a multiobjective optimisation problem, where all the objective functions are tried to be minimised simultaneously. The aim of this paper is to present a new method to solve reconciliation similar to (MDP), using some results of multiobjective optimisation.

2 Methods

In general multiobjective optimisation problems (MOP) optimise more than one objective functions $F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ while $\mathbf{x}$ is in a feasible solution set $\mathcal{F} \subseteq \mathbb{R}^n$. The problem can be formulated in the following way:

$$\mathrm{MIN}_{\mathbf{x} \in \mathcal{F}} F(\mathbf{x}) \quad (MOP),$$

where MIN means that we minimise all the objective functions simultaneously. The optimum values we are looking for in this case are called Pareto-optimal solutions. We call $\mathbf{x}^*$ a (weakly) Pareto-optimal solution of the problem (MOP) if there does not exist a feasible solution $\mathbf{x} \in \mathcal{F}$ which satisfies the vector inequality $F(\mathbf{x}) < F(\mathbf{x}^*)$. In other words $\mathbf{x}^*$ is a Pareto-optimal solution of (MOP) if none of the objective functions can be decreased without increasing some of the other objective functions.
Let $\mathbf{x} \in \mathcal{F}$ be a feasible solution of (MOP). Vector $\mathbf{q} \in \mathbb{R}^n$ is a feasible joint decreasing direction from $\mathbf{x}$ if there exists a $h_0 > 0$ for every $h \in]0, h_0]$ satisfying the followings:

1) $\mathbf{x} + h\mathbf{q} \in \mathcal{F}$,

2) $F(\mathbf{x} + h\mathbf{q}) < F(\mathbf{x})$.

In other words $\mathbf{q}$ is a feasible joint decreasing direction from point $\mathbf{x}$ if we can step from that point in direction $\mathbf{q}$ while all the objective functions are decreasing.
Some methods are known [3] which calculate a joint decreasing direction for a given feasible point of (MOP), or give a proof of the fact that the point is a Pareto-optimal solution of the problem. Based on these results, methods to find Pareto-optimal solution where all the objective functions are smaller than the given feasible solution also have been developed.
Now we are ready to formulate the multiobjective approach of the multivariate Denton problem (MMD). The feasible solution set of the problem is the same as it was in (MDP):

$$\mathcal{F}_{(MMD)} = \{\mathbf{y}^\top = (\mathbf{y}_0^\top, \mathbf{y}_1^\top, \ldots, \mathbf{y}_k^\top) \in \mathbb{R}^{T \cdot (k+1)} | A\mathbf{y}_i = \tilde{\mathbf{y}}_i \quad \forall i; \ \sum_{i=1}^{k} \mathbf{y}_i - \mathbf{y}_0 = \mathbf{0}\}.$$

The problem itself can be formulated in the following way:

$$\text{MIN}_{\mathbf{y}\in\mathcal{F}_{(MMD)}}(\mathbf{y}_i - \mathbf{z}_i)^\top M(\mathbf{y}_i - \mathbf{z}_i) \quad i = 0, 1, 2, \ldots, k \qquad (MMD).$$

We note that the optimal solution of (MDP) is one of the Pareto-optimal solutions of (MMD).

Let us summarise how the official statistics typically faces with reconciliation problems, and how can the results mentioned above be used to solve these problems. At the first time when reconciliation should be made a Pareto-optimal solution of (MMD) can be found by solving (MDP) (other solution for this first step exists). Since the problem works with time series, new data appears time to time, or some revision of the data is also possible. From our approach it means that after we already have a temporal disaggregated solution $\mathbf{y}_{\text{old}}$, the $\tilde{\mathbf{y}}$ and/or $\mathbf{z}$ parameters of (MMD) are changing therefore, we have a new (MMD) to solve. In order to solve this problem we can use the following algorithm:

Algorithm 1 Pseudo code for reconciliation

procedure MULTIOBJECTIVE_SOLVER((MMD), $\mathbf{y}_{\text{old}}$)
    if $\mathbf{y}_{\text{old}} \in \mathcal{F}_{(MMD)}$ **then**
        $\mathbf{y} = \mathbf{y}_{\text{old}}$
    else Let $\mathbf{y}$ be the nearest point to $\mathbf{y}_{\text{old}}$ from $\mathcal{F}_{(MMD)}$
    while $\mathbf{y}$ is not a Pareto-optimal solution of (MMD) **do**
        Calculate feasible joint decreasing direction $\mathbf{q}$ and a proper h.
        $\mathbf{y} = \mathbf{y} + h\mathbf{q}$
return $(\mathbf{y})$

The main difference between this approach and the original multivariate Denton method is that here the $\mathbf{y}_{\text{old}}$ is used. In this algorithm we modified the old solution ($\mathbf{y}_{\text{old}}$) to decrease all the objective functions, while the original approach solves the optimisation problem independently of the old result.

The details of the algorithm and numerical results will be presented in the paper.

References

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